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THE MOTION OF A VISCOUS FLUID IN A PIPE OF FINITE LENGTH†

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A solution is obtained of a linearized, initial-boundary-value problem of the unsteady motion of a viscous fluid in a cylindrical pipe of circular cross-section with a horizontal axis, under the assumption that the pressure at one end of the pipe is constant, and at the other end a device is set up which changes the flow rate of the fluid. It is shown that, depending on the relations between the defining parameters of the problem, the solution tends asymptotically either to a periodic solution, or to one of the two stable steady-state solutions. These results are obtained using methods similar to those developed in [1–3], where the unsteady motions of ground waters during the irrigation were considered, taking into account the evaporation from the regions between the channels. It was assumed there that the irrigation, which was carried out at a constant rate, was switched on and off depending on the level of ground waters in the inspection well.

THE EQUATIONS of unsteady motions of a viscous fluid in pipes, taking hydraulic resistance into account, were derived in [4] for laminar, as well as turbulent flows. Using the assumptions that the pressure at one end of the pipe is a known function of time and that the other end is fitted with a device which changes the flow rate according to a prescribed, time-dependent law (a piston pump, a sluice valve, a turbine or a compressor), and the device is fitted to the pipe either directly, or through a chamber used as a flow-rate regulator or as a pressure oscillations damper (an air chamber, equalizing chamber or receiver), Charnyi as well as Rozenberg and Buyanovskii solved a large number of specific problems. Here either it was assumed that there was no motion, or that the motion was steady when $t \leqslant 0$, or periodic motions were considered.

1. We shall consider the unsteady motion of a viscous incompressible fluid of density ρ, in a pipe of finite length l of circular cross-section, with a horizontal axis. At one end of the pipe ($x = 0$) the pressure is constant (equal, in particular, to the atmospheric pressure) and a device mentioned earlier is attached to the other end ($x = l$), and at the initial instant of time the values of velocity u and pressure p within the pipe are given. We assume that when a chamber is present within the device the flow rate q from the device is a Rayleigh-type function of velocity u, with a non-singlevalued section $u_{**} < u < u_*$ (Fig. 1):

$$q(u) = \begin{cases} q_1 & (u < u_*) \\ q_2 & (u > u_{**}) \end{cases} \qquad (q_1 > q_2) \tag{1.1}$$

Moreover, when $u = u_*$ and $u = u_{**}$, we have jumps from one branch of the function to the other, as shown in Fig. 1. When there is no chamber, the flow rate q (which is identical in this case with the velocity u at $x = l$ is a function of time t:

$$q(t) = \begin{cases} q_1 (n\bar{T} < t < \bar{T}_1 + n\bar{T}) \\ q_2 (\bar{T}_1 + n\bar{T} < t < (n+1)\bar{T}) \end{cases} \qquad (n = 0, 1, 2, \ldots) \tag{1.2}$$

(henceforth, in some of the cases we shall assume that relation (1.2) also holds when a chamber is present).

We shall agree, to fix our ideas, that if $u_{**} < u < u_*$ at the initial instant of time $t = 0$, then $q = q_1$.

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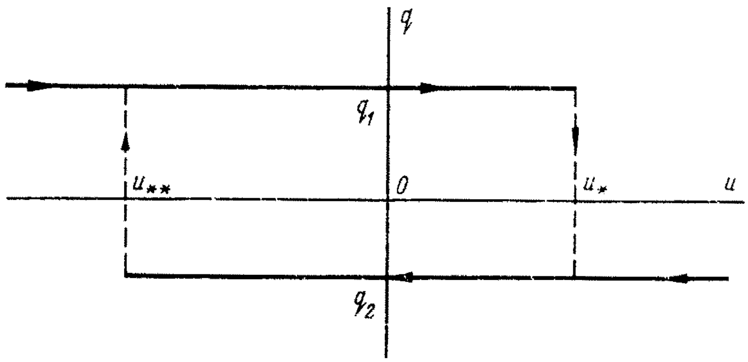

FIG. 1.

We shall also assume that the velocity u is much less than the velocity of sound in the pipe, which is important in the case of a liquid.

The study of the motion reduces to finding, in the region $(0<x<l,\ t>0)$, a solution of the initial-boundary-value problem for the system of two quasilinear partial differential equations quoted in [4].

Eliminating from this system the pressure p and changing to dimensionless coordinates

$$z=x/l,\quad \tau=ct/l,\quad U=u/c \tag{1.3}$$

we reduce the problem to that of solving, in the region $(0<z<1,\ \tau>0)$, a single non-linear, second-order partial differential equation

$$\frac{\partial^2 U}{\partial \tau^2}=\frac{\partial^2 U}{\partial z^2}-A\left(U\frac{\partial^2 U}{\partial z\,\partial \tau}+\frac{\partial U}{\partial z}\frac{\partial U}{\partial \tau}\right)-\sigma(U)\frac{\partial U}{\partial \tau} \tag{1.4}$$

describing one-dimensional motion.

The initial and boundary conditions are, respectively,

$$U(z,\ 0)=U_0(z),\quad \partial U(z,\ 0)/\partial\tau=V_0(z) \tag{1.5}$$

$$\partial U(0,\ \tau)/\partial z=0,\quad b\partial U(1,\ \tau)/\partial z+U(1,\ \tau)=c^{-1}q[U(1,\ \tau),\ \tau] \tag{1.6}$$

The function $q(U,\tau)$ is given by the formulas (1.1) or (1.2), $b>0$ is a dimensionless parameter characterizing the type of chamber, and when there is no chamber $b=0$.

In the case of a pipe of circular cross-section and turbulent flow, we have $A=2$, $\sigma(U)=\zeta U$, and in the case of laminar flow $A=\frac{4}{3}$, $\sigma(U)=2\nu$; ζ and $\nu>0$ are dimensionless parameters depending on the type of flow and the properties of the pipe [4]. The functions $U_0(z)$ and $V_0(z)$are assumed to satisfy the Dirichlet conditions.

Assuming from the isotropic nature of the flow that $U=\varepsilon U_1+\varepsilon^2 U_2$ where $\varepsilon>0$ is a small dimensionless parameter (we can, for example, assume that $\varepsilon=(q_1+q_2)/(2c)$ or $\varepsilon=q_1/c$) and restricting ourselves to the first approximation, we obtain from (1.4) the following linear equations for the function U_1:

$$\partial^2 U_1/\partial\tau^2=\partial^2 U_1/\partial z^2 \tag{1.7}$$

$$\partial^2 U_1/\partial\tau^2+2\nu\partial U_1/\partial\tau=\partial^2 U_1/\partial z^2 \tag{1.8}$$

Equation (1.7) holds for turbulent and (1.8) for laminar flow.

Charnyi assumed in [4] that when the inequality $\nu\geq\pi/2$ is satisfied the telegraph equation (1.8) can be replaced by the heat conduction equation

$$2\nu\partial U_1/\partial\tau=\partial^2 U_1/\partial z^2 \tag{1.9}$$

As usual, we shall require that the function U_1 must satisfy conditions (1.6)

$$\partial U_1(0,\ \tau)/\partial z=0,\quad b\partial U_1(1,\ \tau)/\partial z+U_1(1,\ \tau)=Q[U_1(1,\ \tau),\ \tau] \tag{1.10}$$

$$(Q=q/(\varepsilon c))$$

We shall assume that Eqs (1.7)–(1.9) must be solved using the initial conditions (1.5) and boundary conditions (1.10), where U is replaced by U_1 (the second condition of (1.5) for (1.9) is missing).

Thus we have formulated for the function U_1 (of the first approximation) three initial-boundary-value problems (henceforth we shall restrict ourselves to solving these three problems).

Henceforth we shall call every problem of solving Eqs (1.7), (1.8) and (1.9) in the region ($0<z<1$, $\tau>0$), with conditions (1.5) and (1.10), the first, second and third problem respectively, and one with conditions (1.10), the problem without initial conditions.

We note that after solving each of these three problems defining the first approximation U_1, we can find the second approximation U_2 as a solution of some linear problem with zero initial and boundary conditions, assuming in the latter that $\nu = 0$.

Using the Fourier method, solving the regular Sturm–Liouville problem and assuming that

$$U(z,\tau)=\begin{cases} u_1^{(i+1)}(z,\tau) & (S_i\leqslant\tau\leqslant S_{i,i+1}) \\ u_2^{(i+1)}(z,\tau) & (S_{i,i+1}\leqslant\tau\leqslant S_{i+1}) \end{cases} \tag{1.11}$$

$$S_i=\sum_{j=0}^{i}T^{(j)},\quad S_{i,i+1}=S_i+T_1^{(i+1)},\quad i=0,1,2,\ldots,T^{(0)}=0$$

we find

$$u_1^{(i+1)}=Q_1+\exp[-\nu(\tau-S_i)]\times \tag{1.12}$$

$$\times\sum_{k=1}^{\infty}[C_{1k}^{(i+1)}\cos\omega_k(\tau-S_i)+C_{2k}^{(i+1)}\sin\omega_k(\tau-S_i)]\cos\mu_k z$$

$$u_2^{(i+1)}=Q_2+\exp[-\nu(\tau-S_{i,i+1})]\times \tag{1.13}$$

$$\times\sum_{k=1}^{\infty}[D_{1k}^{(i+1)}\cos\omega_k(\tau-S_{i,i+1})+D_{2k}^{(i+1)}\sin\omega_k(\tau-S_{i,i+1})]\cos\mu_k z$$

Here μ_k ($k = 1, 2, \ldots$) are the roots of the transcendental equation

$$\operatorname{ctg}\mu=b\mu \tag{1.14}$$

$\omega_k = \sqrt{(\mu_k^2-\nu^2)}$ if $\mu_k>\nu$. If $\mu_k<\nu$, then $\cos\omega_k x$ and $\sin\omega_k x$ in (1.12) and (1.13) must be placed by $\operatorname{ch}\sigma_k x$ and $\operatorname{sh}\sigma_k x$ respectively, where $\sigma_k = \sqrt{(\nu^2-\mu_k^2)}$.

We will assume in formulas (1.11)–(1.13) to fix our ideas, that $Q_i = Q_1$ when $\tau = 0$ (it is clear that this restriction is not essential).

The coefficients $C_{1k}^{(1)}$ and $C_{2k}^{(1)}$ are found from the initial conditions (1.5), and the remaining constants $-D_{jk}^{(i+1)}$ ($i = 0, 1, 2, \ldots$), $C_{jk}^{(i+1)}$ ($i = 1, 2, \ldots$), $j = 1, 2$ are determined successively from the conditions for matching the functions and their derivatives in τ at some instants of time, namely

$$u_2^{(i+1)}(z,S_{i,i+1})=u_1^{(i+1)}(z,S_{i,i+1});\quad \frac{\partial u_2^{(i+1)}(z,S_{i,i+1})}{\partial\tau}=\frac{\partial u_1^{(i+1)}(z,S_{i,i+1})}{\partial\tau}$$

$$u_1^{(i+2)}(z,S_{i+1})=u_2^{(i+1)}(z,S_{i+1});\quad \frac{\partial u_1^{(i+2)}(z,S_{i+1})}{\partial\tau}=\frac{\partial u_2^{(i+1)}(z,S_{i+1})}{\partial\tau}$$

When $b = 0$ we have in (1.11), by virtue of (1.2),

$$T^{(j)}=T,\quad T_1^{(i)}=T_1\ (i,j=1,2,\ldots),\quad T=c\bar{T}/l,\quad T_1=c\bar{T}_1/l$$

If $b>0$, the constants $T_1^{(i+1)}$ and $T_2^{(i+1)} = T^{(i+1)} - T_1^{(i+1)}$ will be found successively from the conditions (Figs 1 and 2)

$$u_1^{(i+1)}(1,S_{i,i+1})=U_*,\quad u_2^{(i+1)}(1,S_{i+1})=U_{**}\quad (U_*=u_*/\varepsilon c,\ U_{**}=u_{**}/\varepsilon c)$$

as the smallest roots of these equations.

The above conditions show that cases are possible in which the value U_* or U_{**} will not be reached over a finite interval of time (or, in an infinitely long time interval these values will be reached only a finite number of times).

2. Using relations (1.12)–(1.14) and the similar solution of the third problem (which we shall not write out here), we can see that all three problems without the initial conditions will have Lyapunov-stable steady solutions (while the second and third problem will also have asymptotically stable steady solutions)

$$U(z,\tau)=Q_1,\quad U(z,\tau)=Q_2 \tag{2.1}$$

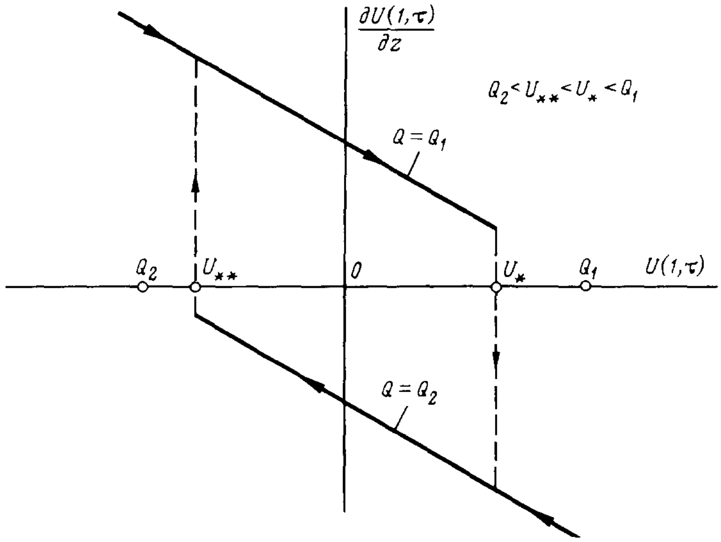

FIG. 2.

Using relations (1.12)–(1.14) and Fig. 2 we can confirm that when $b = 0$ and when $b > 0$ and condition (1.2) holds, the solution of each of three problems tends asymptotically to some limit solution. For the second and third problem the solution is periodic, and for the first problem it represents the sum of the solution (1.12) where $i = Q_1 = \nu = 0$, and a periodic solution, which we shall write out presently, assuming that its period is equal to T and using its following two representations: (1) one similar to (1.12)–(1.14) where we have put $\nu = 0$, and (2) one obtained directly in the form of a Fourier series uring (1.7), (1.10) and (1.2) where $n = 0, \pm 1, \pm 2, \ldots$; if on the other hand the results hold for the second and third problem and the inequality $b > 0$ holds, provided that condition (1.1) holds and

$$Q_2 < U_{**} < U_* < Q_1 \tag{2.2}$$

The first of these representations has the form

$$u_1(z,\tau) = Q_1 + \sum_{k=1}^{\infty} (C_{1k}\cos\mu_k\tau + C_{2k}\sin\mu_k\tau)\cos\mu_k z \qquad (0 \leqslant \tau \leqslant T_1) \tag{2.3}$$

$$u_2(z,\tau) = Q_2 + \sum_{k=1}^{\infty} [D_{1k}\cos\mu_k(\tau - T_1) + D_{2k}\sin\mu_k(\tau - T_1)]\cos\mu_k z \qquad (T_1 \leqslant \tau \leqslant T)$$

$$C_{1k} = -a_k g_k \cos(^1/_2)\mu_k T_1, \quad C_{2k} = -a_k g_k \sin(^1/_2)\mu_k T_1$$

$$D_{1k} = a_k + C_{1k}, \quad D_{2k} = -C_{2k}$$

$$g_k = \frac{\sin(^1/_2)\mu_k(T - T_1)}{\sin(^1/_2)\mu_k T}, \quad a_k = \frac{2(Q_1 - Q_2)\sin\mu_k}{\mu_k[1 + \sin 2\mu_k/(2\mu_k)]}$$

The second representation is given by the relation

$$U(z,\tau) = \bar{U}(z,\tau) = \frac{Q_1 T_1 + Q_2(T - T_1)}{T} + \sum_{n=1}^{\infty} \frac{(c_n\cos\alpha_n\tau + d_n\sin\alpha_n\tau)}{(\cos\alpha_n - b\alpha_n\sin\alpha_n)}\cos\alpha_n z, \quad \alpha_n = \frac{2\pi n}{T} \tag{2.4}$$

$$c_n = \frac{(Q_1 - Q_2)}{\pi}\,\frac{(1 - \cos\alpha_n T_1)}{n}, \quad d_n = \frac{(Q_1 - Q_2)}{\pi}\,\frac{\sin\alpha_n T_1}{n}$$

(Here we assume that $b \geqslant 0$ and the boundary condition (1.2) is used.)

In the case of resonance formulas (2.3) and (2.4) must be modified.

Let the relation $\operatorname{ctg}\alpha_s = b\alpha_s$ hold when $n = s$. Then for any $k = m$ we have $\alpha_s = \mu_m$ or $T = 2\pi s/\mu_m$, and $\sin(1/2)\mu_m T = 0$ and the terms containing $k = m$ and $n = s$ on the right-hand sides of formulas (2.3) and (2.4) respectively, become infinite. (It can be shown that if $T_1 = T/2$, $s = 2r$, then the singularity will be removed.)

For the time being we shall assume, for simplicity, that formulas (2.3) and (2.4) contain only a single term containing $k = m$ and $n = s$ (since the problem is linear, the restriction is not important).

Using the first formula of (2.3) and (1.14), we obtain

$$L[u_1(1,\tau)] = b\partial u_1(1,\tau)/\partial z + u_1(1,\tau) =$$

$$= Q_1 + \sum_{k=1}^{\infty} (C_{1k}\cos\mu_k\tau + C_{2k}\sin\mu_k\tau)(\cos\mu_k - b\mu_k\sin\mu_k) = \tag{2.5}$$

$$= Q_1 - a_m A_m \sin(1/2)\mu_m(T-T_1)\cos(1/2)\mu_m(\tau-T_1)$$

$$A_m = (\cos\mu_m - b\mu_m\sin\mu_m)/\sin(1/2)\mu_m T$$

Using the l'Hopital's rule to expand the indeterminancy we find that the quantity A_m is finite and non-zero for all $b \geq 0$. It can be shown that $L[u_2(1,\tau)] = L[u_1(1,\tau)] + Q_2 - Q_1$, and it follows from (2.5) that

$$b\partial U(1,\tau)/\partial z + U(1,\tau) = \tag{2.6}$$

$$= Q[U(\tau),\tau] - a_m A_m \sin(1/2)\mu_m(T-T_1)\cos(1/2)\mu_m[\tau - (1/2)T_1]$$

The second term on the right-hand side of (2.6) taken with the minus sign, represents the sth term of the expansion of the quantity $b\partial \bar{U}(1,\tau)/\partial z + \bar{U}(1,\tau)$ where the function $\bar{U}(z,\tau)$ is given by the formula (2.4), i.e. $b\partial \bar{u}_s(1,\tau)/\partial z + \bar{u}_s(1,\tau)$.

Thus, from (2.6) and (1.10) it follows that we have obtained a periodic solution of the second problem with boundary conditions (1.10), the term corresponding to the resonant term with $k = m$ in formula (2.3) and with $n = s$ in formula (2.4), is removed. Therefore in order to obtain the solution of the problem in question we must replace this term in (2.3) and (2.4) by $w(z,\tau) = \bar{u}_s(z,\tau)$, i.e. by the solution of the first problem without initial conditions, with boundary conditions

$$\partial w(0,\tau)/\partial z = 0, \quad b\partial w(1,\tau)/\partial z + w(1,\tau) = B_s(\tau)$$
$$(B_s(\tau) = b\partial \bar{u}_s(1,\tau)/\partial z + \bar{u}_s(1,\tau) = c_s\cos\mu_m\tau + d_s\sin\mu_m\tau) \tag{2.7}$$

By virtue of the first condition of (2.7) we find that $w(z,\tau) = f(\tau+z) + f(\tau-z)$ where $f(x)$ is the unknown function, and from the second condition of (2.7) we obtain for it the following differential-functional equation:

$$b[f'(\tau+1) - f'(\tau-1)] + f(\tau+1) + f(\tau-1) = B_s(\tau) \tag{2.8}$$

whose solution, by virtue of the second condition of (2.7), has the form

$$f(\tau) = -(1/2)\tau B_s'(\tau)\{\mu_m[(b+1)\sin\mu_m + b\mu_m\cos\mu_m]\}^{-1}$$

Thus the periodic solution is replaced, in the case of resonance, by a solution containing secular terms.

Similar results are known for other problems (see e.g. [5]).

It can be shown that there is no resonance in the solution of the second problem without initial conditions.

Formulas similar to (2.3) and (2.4), describing the solutions of the second and third problem without initial conditions and without a chamber ($b = 0$), also determine the forced oscillations (in which case the influence of the initial conditions vanishes). As for solving the first problem without initial conditions for the case of $b \geq 0$ and in the case of finding the forced oscillations we must add to the solution (2.3), as we mentioned above, the solution (1.12) $u_1^{(')}(z,\tau)$ of the first problem (where we put $\nu = Q_1 = 0$).

Comparing the solutions of the first and second problem we see that when $\nu \to 0$, then the condition of "vanishing friction", introduced for other problems [6], holds.

When a chamber is in place ($b > 0$), the formulas similar to (2.3)–(2.4) for the second problem, describe the self-excited oscillations which occur under the conditions (2.2), and formulas (1.11)–(1.14) describe the flow for given initial conditions for any instant of time, i.e. the process by which these self-excited oscillations appear.

If conditions (2.2) do not hold (for example if $Q_2 < U_{**} < Q_1 < U_*$, as shown in Fig. 3), then cases are possible in which self-excited oscillations do not appear. For example, if $\nu > 0$, then for some values of the functions $U_0(z)$ and $V_0(z)$ $(0 < z < 1)$ appearing in the initial conditions (1.5), the motion will tend asymptotically to a steady state described by the first relation of (2.1).

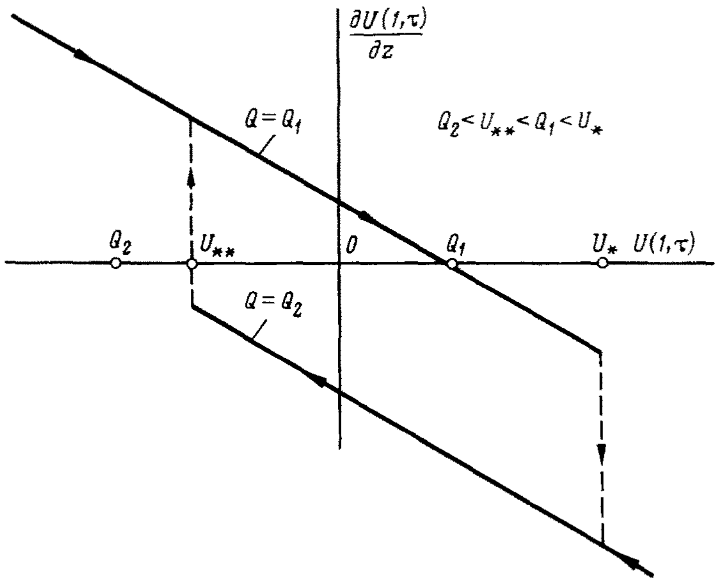

FIG. 3.

Example. Let $U_0(z) = C = \text{const}$, $V_0(z) = 0$. Then formula (1.12) in which we put $i = 0$, will take the form

$$u_1^{(1)}(z,\tau) = Q_1 + Pe^{-\nu\tau}\sum_{k=1}^{\infty} p_k\left(\cos\omega_k\tau + \frac{\nu}{\omega_k}\sin\omega_k\tau\right)\frac{\cos\mu_k z}{\cos\mu_k}$$

$$(P = 2b^{-1}(C-Q_1),\quad p_k = \cos^2\mu_k(b\mu_k^2+\cos^2\mu_k)^{-1}) \tag{2.9}$$

If the inequality

$$C < Q_1 + \frac{b}{2}(U_* - Q_1)\left[\sum_{k=1}^{\infty} p_k + \nu\sum_{k=1}^{\infty}\frac{p_k}{\omega_k}\right]^{-1}$$

holds, then from (2.9) we see that the value of U_* will never be attained by the function $u_1^{(1)}(1,\tau)$.

The numerical values of the first seven parameters μ_k and ω_k appearing in (2.9) are given below for $b = 3.3446605 \times 10^{-2}$, $\nu = 1.4$:

k	1	2	3	4	5	6	7
μ_k	1.52	4.561	7.605	10.653	13.707	16.769	19.833
ω_k	0.552	4.341	7.475	10.561	13.636	16.710	19.784

Using the methods developed earlier in [2, 3] we can confirm that the results obtained for the third problem are completely analogous to the results obtained above for the first and second problem; in particular, there is no resonance.

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